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**Article** *in* Journal of Hydrology · September 1988

DOI: 10.1016/0022-1694(88)90090-X

*Journal of Hydrology*, 102 (1988) 29–47
Elsevier Science Publishers B.V., Amsterdam — Printed in The Netherlands



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# EFFECTS OF SPATIAL VARIABILITY AND SCALE WITH IMPLICATIONS TO HYDROLOGIC MODELING

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(Received April 29, 1987; accepted after revision September 9, 1987; received at publisher March 9, 1988)

ABSTRACT

Wood, E.F., Sivapalan, M., Beven, K. and Band, L., 1988. Effects of spatial variability and scale with implications to hydrologic modeling. In: R.L. Bras, M. Hino, P.K. Kitanidis and K. Takeuchi (Editors), Hydrologic Research: The U.S.–Japan Experience. J. Hydrol., 102: 29–47.

This paper reports the results of a preliminary investigation into the existence of a Representative Elementary Area (REA) in the context of hydrologic modeling at the catchment scale. The investigation was carried out for an actual catchment topography as represented by Coweeta River experimental basin with synthetic realizations for rainfall and soils. The hydrologic response of this catchment was modeled by a modified version of TOPMODEL* which is capable of modeling both infiltration excess and saturation excess runoff and incorporating the spatial variability of soils, topography, and rainfall. The effect of scale was analyzed by first dividing the catchment into smaller subcatchments and determining the average water fluxes for each subcatchment. The preliminary results lead to the following conclusions: (1) a Representative Elementary Area (REA) exists in the context of catchment hydrologic responses; (2) the REA is strongly influenced by the topography; and (3) based on our initial results, the length scale of rainfall seems to have only a secondary role in determining the size of the REA; however, increases in the variability of rainfall and soils between subcatchments increase the variability of runoff generation between subcatchments.

## 1. INTRODUCTION

The influence of catchment scale on hydrologic response and its importance in rainfall–runoff modeling has been recognized since the early 1960s (Minshall, 1960; Amorocho, 1961). Qualitatively it has been recognized that as the spatial scale of the catchment increases, the catchment tends to attenuate the complex, local patterns of runoff generation and water fluxes. As pointed out by Amorocho (1961), at large catchment scales the runoff generation becomes somewhat insensitive to rainfall intensity changes recorded at

*Beven and Kirkby (1979)

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individual gages and the catchment-scale rainfall–runoff response appears to be governed by macroscale catchment characteristics. These observations have stimulated hydrologists' attempts to define what is meant by "large catchment scales" and to develop a consistent theory for analyzing catchment responses at different scales.

The paper by Dooge (1982) addresses the issue of hydrologic parameterization at different scales. In reviewing the different approaches for modeling hydrologic processes at varying scales, Dooge makes the point that linking phenomena at field scales (10–100 ha) and catchment scales (10–1000 $km^2$) is an unresolved problem. This paper addresses this problem and provides some initial results.

## 2. SCOPE OF THE PAPER

This paper is a preliminary attempt to analyze the following question related to spatial heterogeneity and catchment scale: How does the statistical behavior of runoff generation change with increases in catchment scale? In the analysis reported here, different catchment scales are represented by hydrologically consistent subcatchments of the Coweeta River experimental basin in North Carolina operated by the U.S. Forest Service, Southeastern Forest Experimental Station.

The question posed above has at its foundation broader concepts that are critical to the modeling and parameterization of hydrological processes. It is our belief that at small scales actual patterns of topography, soil, and rainfall characteristics are important in governing runoff production. Differences in the actual patterns of variability between areas at this scale will produce different responses, even if the underlying distributions are identical. The actual variabilities within the areas represent different realizations from these distributions. However, as scale increases, more and more of the variability in the distributions is sampled within each area, until eventually at some large scale, all areas will yield almost identical responses for the case of stationary distributions.

However, no real process is ever truly stationary or homogeneous. In geophysical phenomena we nearly always have a relatively rapidly varying quantity superimposed on a slowly varying one (Lumley and Panofsky, 1964). Runoff generation is a multiscale phenomenon, with different length scales characterizing soil, topography, and rainfall variability, each of which may itself have small- and large-scale components. For example the integral scale of soil properties at field scales is never more than 100 m, however at much larger scales there are nonhomogeneities or trends brought about by large-scale geologic formations. The question to be asked then is whether we will be able to form, at a certain scale, some average hydrologic response which is invariant or varies only slowly with increasing catchment area. The criterion for the existence of such an average response that seems most appropriate is that the variability of the responses between different areas should fall to acceptably low values for a sufficiently large area; for larger areas this variability may rise again (Lumley and Panofsky, 1964).

We suggest that this threshold scale represents a "Representative Elementary Area" (REA) which will be a fundamental building block for catchment modeling. The REA is a critical area at which implicit continuum assumptions can be used without knowledge of the patterns of parameter values, although some knowledge of the underlying distributions may still be necessary. The size of the REA is expected to depend crucially on the correlation scales inherent in the rainfall–runoff response, and any nonstationarities present in the catchment parameters or inputs.

At present we have used simulation to analyze and probe the concepts and hypotheses put forth above. The simulations are carried out for a fixed topography as represented by Coweeta River experimental basin; the rainfall and soil properties are stationary synthetic realizations but represent typical conditions. The rest of the paper is as follows: in section 3 a more complete discussion of the concept of a representative elementary area is presented; sections 4 and 5 discuss the hydrologic model and details of the simulation experiment; section 6 gives some results and discussion; and section 7 presents our conclusions and directions for future research.

## 3. REPRESENTATIVE ELEMENTARY AREA (REA)

A catchment can be treated as being composed of numerous (infinite) points where infiltration, evaporation and runoff form the local water balance fluxes. If a continuum representation holds, we can replace the actual catchment with all its heterogeneity in soils, topography, and rainfall inputs with a spatially integrated representative catchment or an assemblage of such representative subcatchments. The hydrologic variables at every location within each such catchment or subcatchment are related to its average value through some associated probability distributions. Each mathematical point in the continuum is associated with the area over which the average values are taken. This averaging area acts as the smallest discernible point which is representative of the continuum. We define this area as the Representative Elementary Area (REA). It is strictly analogous to the concept of the Representative Elementary Volume (REV) in porous media (Bear, 1972; Hassanizadeh and Gray, 1979; Shapiro, 1981; Dagan, 1986). The possible use, at catchment scales larger than a certain threshold value, of simple phenomenological equations to quantify runoff generation, as opposed to deductions based on differential equation formulations at the hydrodynamical scale, has also been suggested by Gupta et al. (1986).

As explained by Shapiro (1981), crucial to the validity of the continuum concept is the size of the averaging area. The following properties must hold if the averaging area is to be a valid REA and for the averages to be meaningful quantities:

(1) According to Hazzanizadeh and Gray (1979) the REA must satisfy the inequality:

$$l \ll D \ll L \tag{1}$$

where $l$ is the length scale characteristic of the rapidly varying components of the hydrologic response and $L$ is the length scale of the slowly varying quantities or of the gross inhomogeneities, and $D$ is the length scale of the REA.

(2) The average values obtained must be independent of the size of the REA or vary only smoothly with increasing size of REA to insure that the values are statistically representative of the continuum (Bear, 1972; Shapiro, 1981).

(3) According to Cushman (1984), the existence of the REA is governed by two hypotheses. The indifference hypothesis states that there exists an ordered triplet of length scales ($l$, $D$, $L$) which remain constant irrespective of which field property is studied. For example, the REA must be the same for runoff production as for rainfall. The invariance hypothesis states that the triplet ($l$, $D$, $L$) remain invariant or at least smoothly varying with time and the location on the field.

(4) The equations associated with the REA scale representation must contain all the "physics" of the sub-REA processes. Thus, one must choose a method of averaging the point hydrologic equations to accomplish this result. This averaging will necessarily incorporate some representative patterns of variability of soils, topography, and rainfall. A first step in this direction has been taken by Sivapalan et al. (1987) with their conceptual model of runoff production.

Work reported in this paper is confined in the main to an investigation into the existence of a REA and the associated continuum representation in accordance with points 2 and 3 above. We will not go into the actual averaging of the point scale equations to obtain the REA-scale representation. However, some preliminary results on the averaging of the point scale equations for a simplified case (runoff by infiltration excess only) are included as illustrative examples of the continuum representation that is sought.

## 4. OVERVIEW OF THE ANALYSIS

As stated earlier, in the analysis presented here we have used a simulation approach to investigate the existence of a REA in the catchment hydrologic context. Any such simulation of catchment response will involve the interplay of three important constituents: (1) a catchment that can easily be disaggregated into any number of subcatchments which are themselves natural hydrologic units and not arbitrary shapes; (2) a hydrologic model that can be parameterized at a scale, here termed the point scale, very much smaller than the smallest of the subcatchments such that the average response of every subcatchment or catchment can be considered to be identical to the average of all the point responses within it; and (3) specification of catchment properties (topography, soils, etc.) and inputs (rainfall) at the point scale.

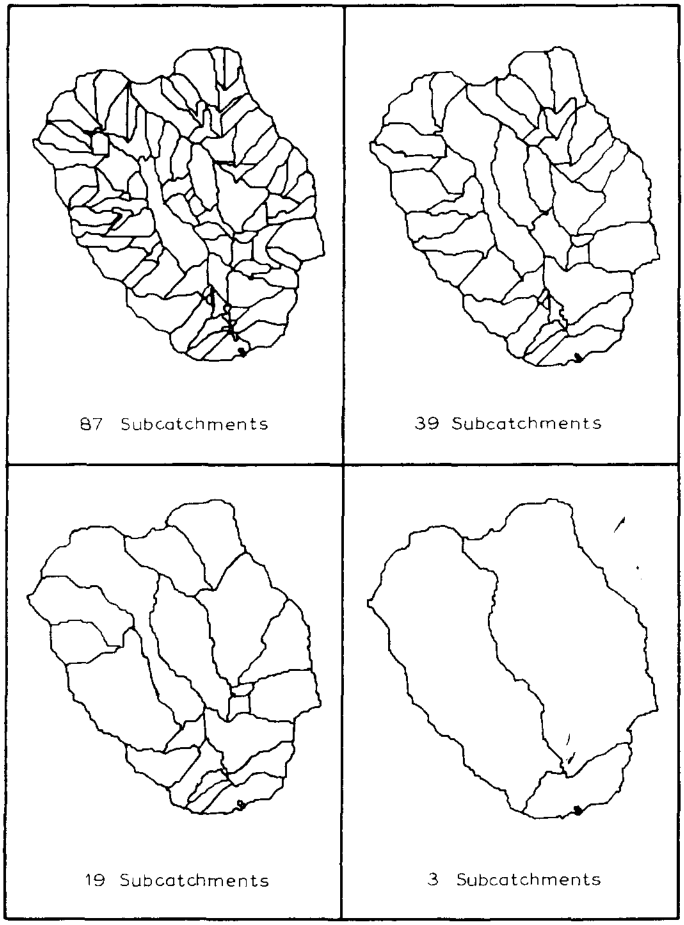



Fig. 1. Natural division of the Coweeta River experimental basin into subcatchments. Division was carried out bv the techniques of Band and Wood (1986).

For the purpose of this study, we used the Coweeta River experimental catchment in North Carolina (area = 17 km$^2$). A digital elevation map (DEM) of topographic elevation data on a 30 m grid is available. The catchment was subdivided at different levels of disaggregation by techniques described by Band and Wood (1986). Four levels of disaggregation were used resulting in the subdivision of the catchment into 87, 39, 19, and 3 subcatchments in that order. Figure 1 shows that these subcatchments are made up of a variety of shapes and sizes even at the same level of discretization.

The next step in the analysis is to formulate a hydrologic response model at the pixel (point) scale (30 m$^2$). In this study, we confine our attention to storm response and require a model that could handle what we consider are important processes in storm runoff generation. We used a modified and spatially distributed version of TOPMODEL (Beven and Kirkby, 1979; Beven, 1986). This version of the model is capable of modeling both the infiltration excess (Hortonian) and saturation excess (Dunne) mechanisms of runoff generation with topography-controlled initial soil moisture deficits. The outputs of the model for each pixel are the various water fluxes which can then be aggregated up to determine the values for each subcatchment. Runoff routing is not included in the model.

A fundamental parameter in TOPMODEL is the topographic index $\ln(a/\tan\beta)$ where, for any location on a hillslope, $a$ represents the upslope drainage area per unit contour length and $\tan\beta$ is local ground surface slope. This parameter is used to predict the topographic redistribution of subsurface moisture. Software was written to calculate $\ln(a/\tan\beta)$ at every pixel using the digital elevation data. The extraction of this parameter from the DEM is critical to the usefulness of TOPMODEL as an operational hydrologic model.

Spatially variable topography came from the natural topography. We also wanted to include the effects of variable soils and rainfall inputs on runoff generation. Since soil and rainfall are not available at 30 m intervals we generated stationary random fields of soil hydraulic conductivity and rainfall rates. Appendix B describes their generation. In the initial analyses described here, the generated rainfall rates were spatially variable but temporally constant.

## 5. THE SIMULATION EXPERIMENT

The simulations were carried out in the following manner. The random field for the soils was generated and held constant throughout the analysis. The soil has an average hydraulic conductivity of $0.008325\,\mathrm{cm\,min}^{-1}$, a coefficient of variation of 1.0 and a correlation length, $\lambda_{\ln K_s}$, of 100 m. Next, the rainfall field was generated using the correlogram derived by Sivapalan and Wood (1987) which considers the rainfall field due to a single rainband. In the results presented here we used an average rainfall rate of $0.008325\,\mathrm{cm\,min}^{-1}$, a coefficient of variation of 1.6 and three different correlation lengths $\lambda_p = 125\,\mathrm{m}$, 625 m and 1250 m. The catchment was subdivided into subcatchments at the four levels of discretization mentioned above (Fig. 1). Values of the $\ln(a/\tan\beta)$ were extracted from the DEM data for Coweeta basin. The various soil and rainfall parameter values mentioned here and in Appendices A and B were chosen arbitrarily but are typical values.

With the distributed parameter values of $\ln(a/\tan\beta)$, $K_s$, and $p$, the storm response model mentioned in section 4 above and described in detail in Appendix A is run. The pixel scale outputs from the model (cumulative rainfall, infiltration, and runoff and runoff rate) one hour into the storm are then averaged over the various subcatchments of which there are 148. The simulations and averaging are carried out for five rainfall realizations for each of the three correlation lengths mentioned.

## 6. RESULTS AND DISCUSSION

The results are divided into three groups: (1) variable soils, rainfall, and topography that form, what we consider, a set closest to the natural response of the catchment; (2) spatially constant soil and rainfall with variable topography; and (3) spatially variable soils and rainfall, with runoff production by infiltration excess only. In all cases the point scale responses were averaged over each of the 148 natural subcatchments.

In order to study the behavior of the mean areal response with increasing catchment scale, the following procedure was adopted. The model outputs from all the 148 subcatchments (point-scale outputs averaged over the points within the subcatchments) were arranged in ascending order of increasing subcatchment areas (for convenience represented by the number of pixels). A moving average is taken of this series using a window of fifteen subcatchments, moving in steps of five. The moving averages are then plotted against the average area within the moving window. The variance and standard deviation of the subcatchment responses within each window were also computed. As the averaging area increases, one would expect that the variance of the average storm response would decrease; this is borne out by the simulations. The results of the various experiments are presented in Figs. 2–6.

The results for group 1 are presented in Fig. 2a and b for a rainfall correlation length of 625 m. The solid lines represent the output from the five different rainfall realizations for this case. The unconnected circles is the average over the five realizations. In studying Fig. 2, it is clear that the behavior of the subcatchment responses for average areas under about 1200 pixels (about 1.0 $km^2$) is fundamentally different from the behavior above this size. It shows the stabilization of the mean areal response above an average area (over the windows) equal to about 1.0 $km^2$. This threshold scale was the same for all the outputs studied (infiltration, rainfall, runoff, etc.).

It is important at this stage to ask the question: what are the factors that control the size of the observed REA? Initially it was hypothesized that the size of the REA is controlled by the following factors: (1) the topography of the catchment; and/or (2) the soil/rainfall correlation lengths – all of these in conjunction with the topographically controlled hydrologic response. To determine which factors control the size of the REA, we first carried out the averaging described above on the cumulative rainfall volume. The standard deviations (within the windows) resulting from the above averaging are shown in Fig. 3a. We notice that the behavior is essentially the same as for the infiltration and runoff results shown in Fig. 2. It is our belief that this is due to averaging over the topographically defined subcatchment areas and that these areas which are complex in size and shape have a tremendous effect on the statistics of the cumulative rainfall volume. To demonstrate this, the rainfall was averaged in an analogous way over square areas. As can be seen in Fig. 3b, this averaging produces a smooth variance reduction with area as one would expect. Our conclusion is that the shape of natural subcatchments may affect the REA size.

To probe this further, the initial analysis was simplified by holding both soil and rainfall spatially constant. The resulting variability would then be due to topographic variability only. The results are given in Fig. 4a and b. They show that mean subcatchment response has stabilized at an average area (over the

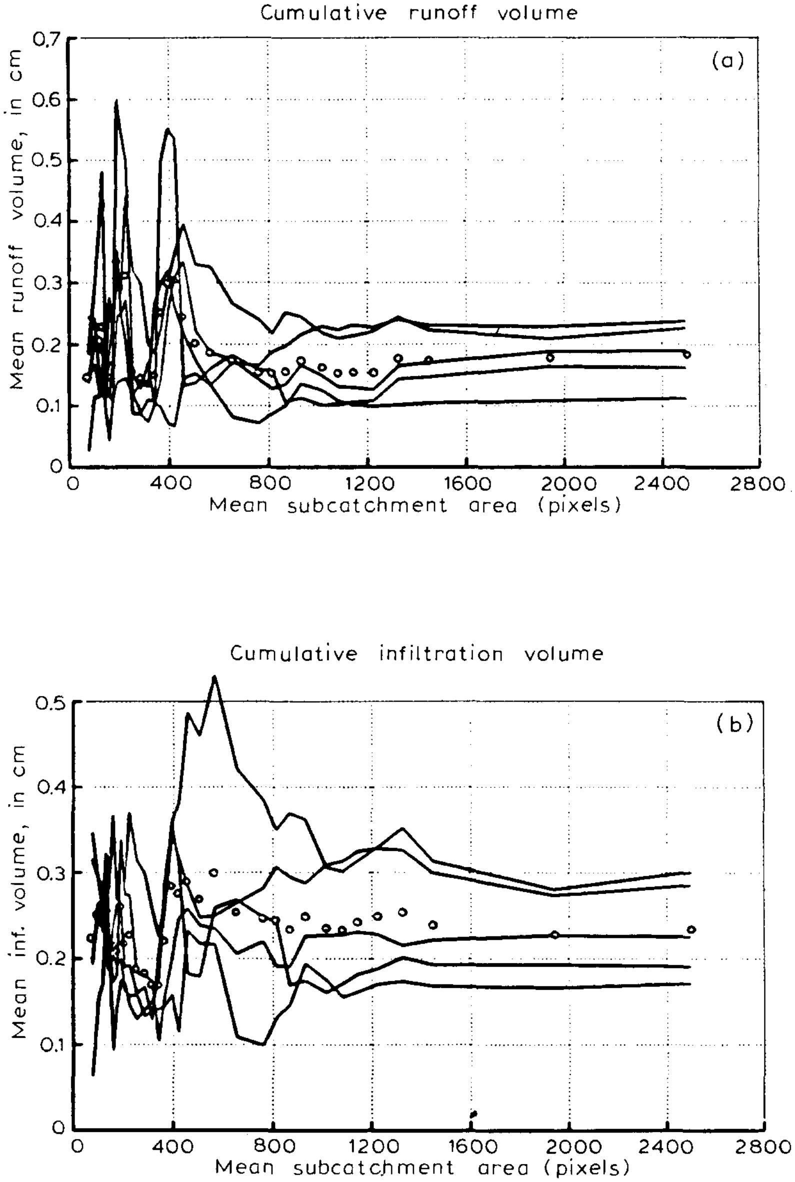
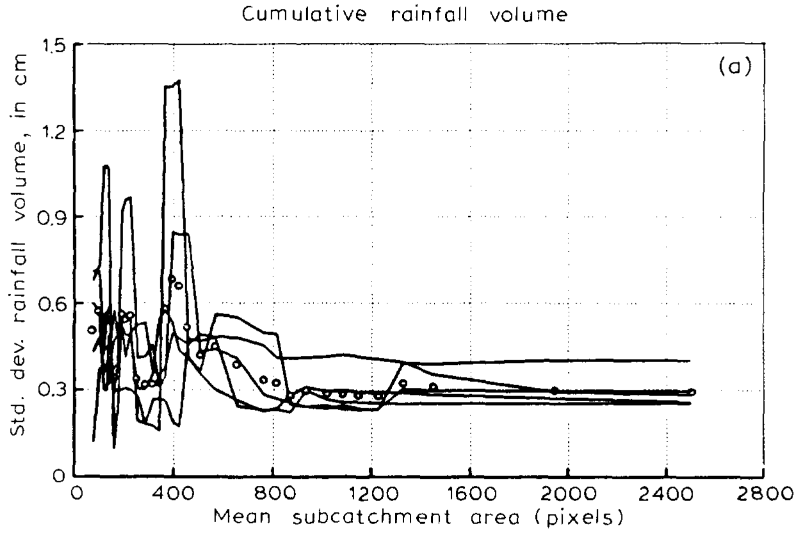



Fig. 2. Mean hydrologic response as a function of mean subcatchment area for five different rainfall realizations. (a) Cumulative runoff volume; (b) cumulative infiltration volume. $\lambda_p$ = correlation length of rainfall = 625 m; circles denote averages over the five realizations.

window) of about 1.0 km$^2$, about the same as before. The variability of the subcatchment responses within the windows was reduced by a factor of 10 as measured by the standard deviation. From the preliminary results presented so far, it appears that although much of the variability between subcatchment



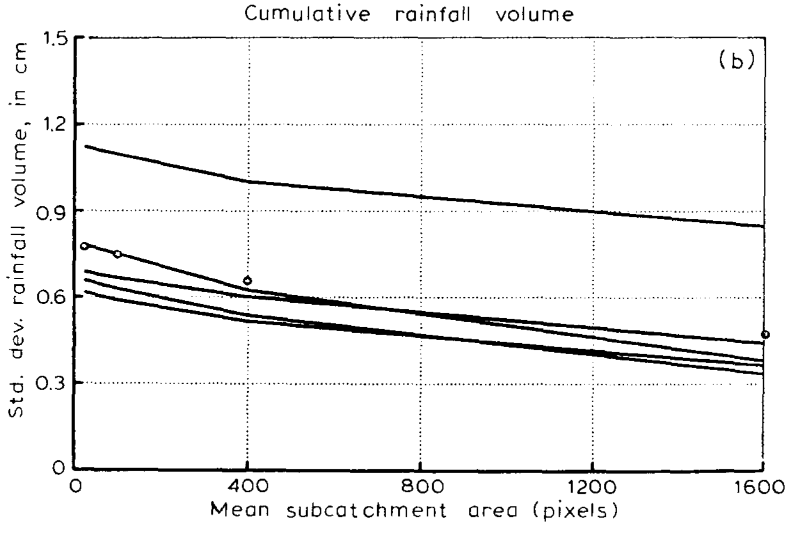
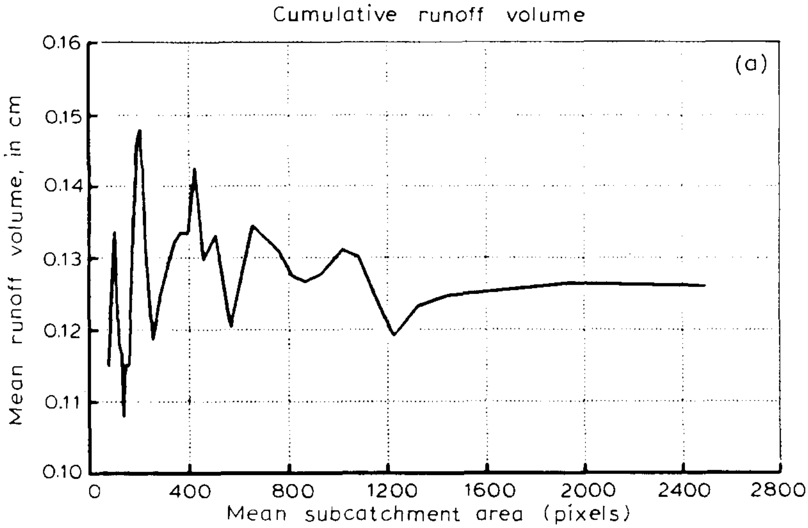




Fig. 3. Standard deviation of cumulative rainfall volume within a moving window as a function of mean subcatchment area. (a) Natural subcatchments; and (b) square subcatchments $\lambda_p = 625\,\mathrm{m}$.

responses is contributed by rainfall and soil variability, the size of the REA is governed primarily by the topography through its role in (a) subcatchment formation and disaggregation and (b) the storm response model, especially the saturation excess runoff generation.



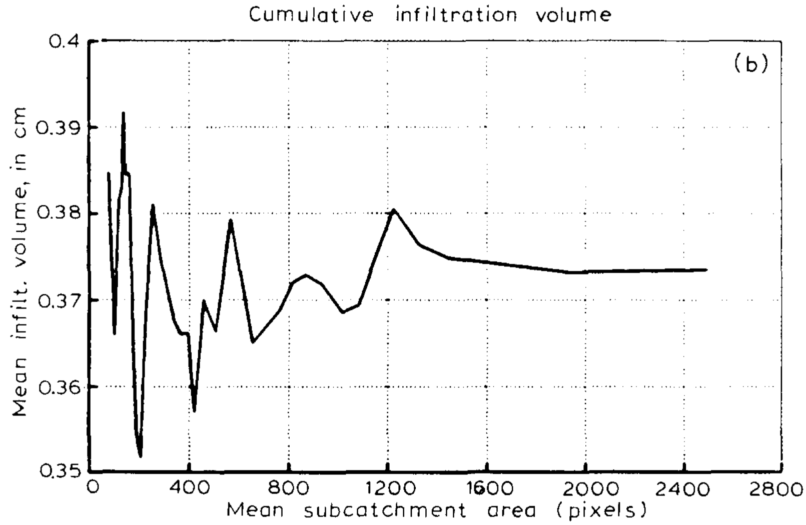
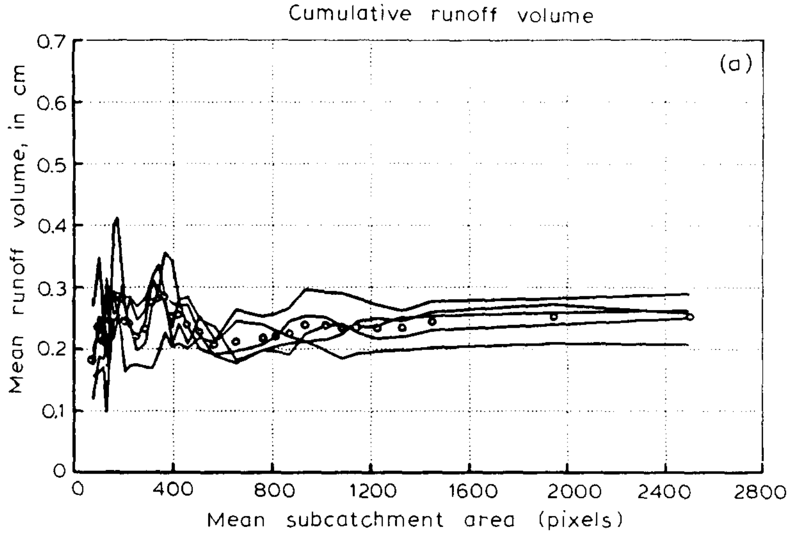




Fig. 4. Mean hydrologic response as a function of mean subcatchment area with spatially constant soils and rainfall. (a) Cumulative runoff volume; (b) cumulative infiltration volume.

The conclusion concerning the role of topography in determining the REA size would imply that keeping the topography fixed and varying the rainfall correlation would have little effect on the REA. To test this out two additional rainfall correlations were analyzed – 125 m and 1250 m. The results for cumulative runoff volume, shown in Fig. 5a and b, support the above conclusions. In fact the impact of decreasing/increasing the rainfall correla-



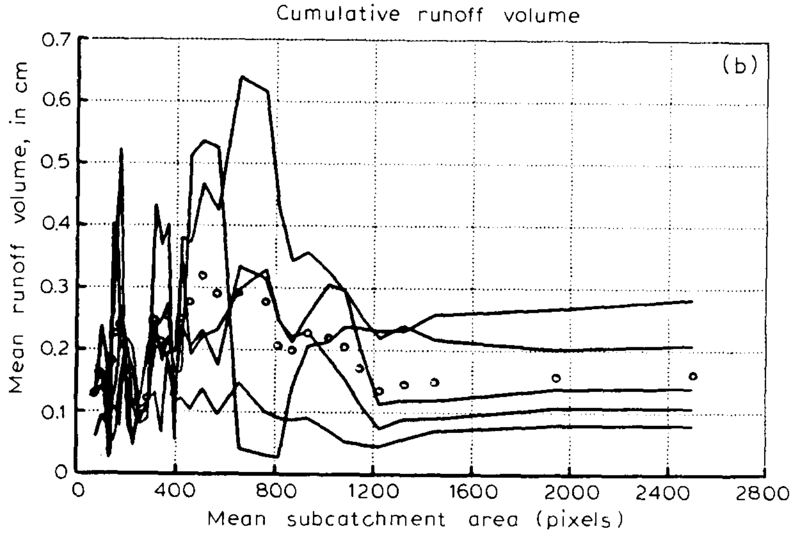
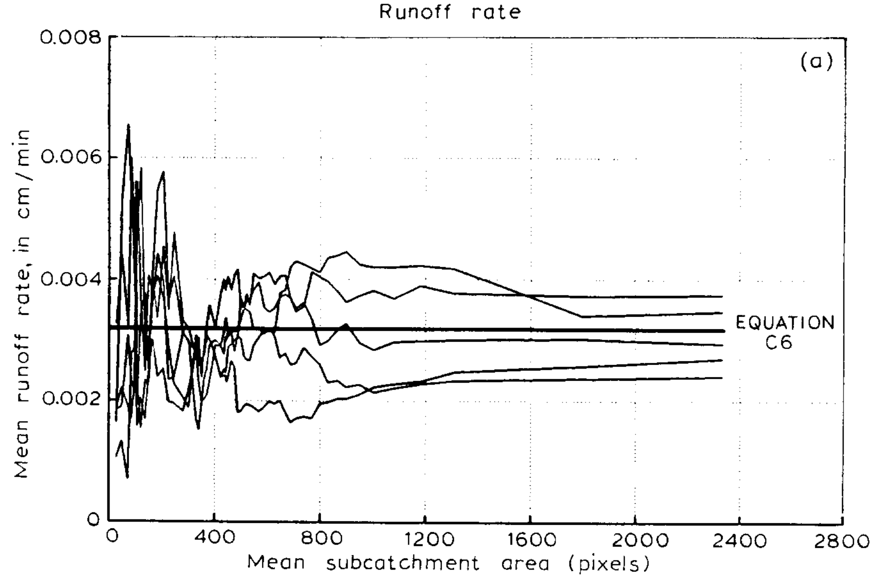




Fig. 5. Areal mean cumulative runoff volume as a function of mean subcatchment area for five different realizations of rainfall. (a) $\lambda_p = 125\,\mathrm{m}$; (b) $\lambda_p = 1250\,\mathrm{m}$.

tion is to decrease/increase the variability within the window and among the five realizations. However the threshold size of the averaging area, the REA, remains relatively unchanged.

Sivapalan (1986) has studied the combined effects of the spatial variability of soils and rainfall on the runoff production by infiltration excess. Quasi-analytical expressions were derived for the statistics of the storm runoff



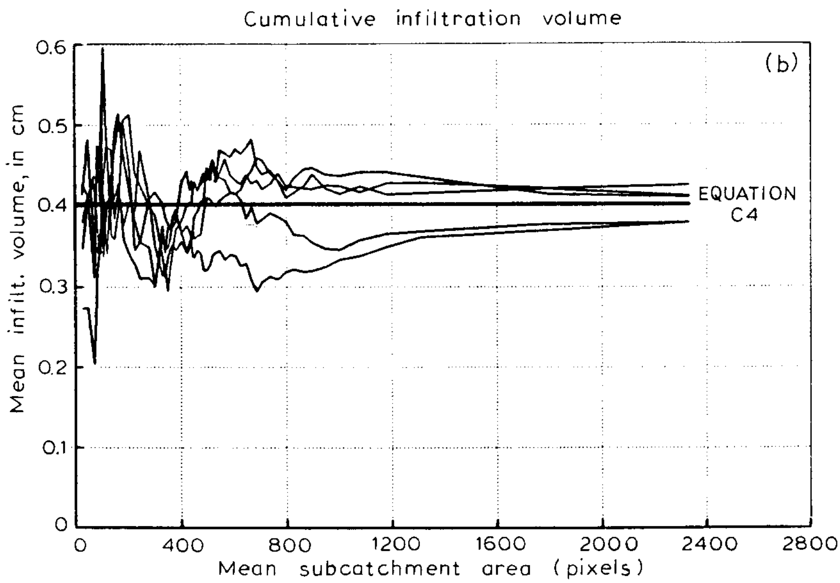




Fig. 6. Comparisons of simulation and analytical results for the case of spatially variable soils and rainfall and runoff production by infiltration excess. (a) Runoff rate and (b) cumulative infiltration volume. $\lambda_p = 625$ m.

response (e.g., ponding time, saturated areas, rates and cumulative volumes of infiltration and runoff). A summary of the resulting analytical expressions are given in Appendix C. The derivations have assumed that the catchment area is large — much larger than the correlation lengths of soil and rainfall spatial variability. Within the REA theory one would then expect these equations to be applicable at catchment scales greater than the REA when runoff generation is by infiltration excess only.

In order to confirm this, the analytical results were compared to results obtained for the Coweeta catchment by simulation for the hypothetical case where runoff is produced by infiltration excess only (Appendix A2). The TOPMODEL formulation described in Appendix A1 will not apply. The "large-scale" model of infiltration excess is as described in Appendix C. The results for runoff rate and cumulative volume of infiltration obtained analytically and by simulation are shown in Fig. 6a and b. It is clear that there is an excellent match between the expected values of the hydrologic responses predicted by simulation and by the analytical expressions of Appendix C. It is our belief that the latter are the relevant equations at the REA scale. This is an important result for there now exists for the first time, for this simple case, a consistent and defendable "large-scale" representation for runoff production fluxes. This result needs to be extended to the case where both mechanisms of runoff generation are operative and to include topographic effects. A preliminary first step in this direction has been made with the work of Sivapalan et al. (1987).

## 7. CONCLUSIONS

The results presented here are initial results obtained from an investigation into the existence of a Representative Elementary Area (REA) in the context of the hydrologic modeling at the catchment scale. The investigation was carried out for an actual catchment topography as represented by Coweeta River experimental basin with synthetic realizations for rainfall and soils.

Based on the preliminary results reported here, the following conclusions can be reached.

(1) A Representative Elementary Area (REA) does exist in the context of the runoff generation response of catchments.

(2) The REA is strongly influenced by the topography, through (a) the sizes and shapes of subcatchments and (b) its role in the hydrologic response model.

(3) Our preliminary results indicate that the variabilities of soils and rainfall inputs between subcatchments have only a secondary role in determining the size of the REA; however, decrease/increase of these variabilities decreases/increases the variability between subcatchments.

We are currently investigating the existence and properties of the REA using a detailed three-dimensional saturated–unsaturated model of the catchment response. We are also looking to repeat the simulations reported here with different catchments and alternate subdivisions. Finally, we are investigating the possibility of averaging the microscale equations of runoff responses over the REA in order to obtain macroscale continuum descriptions of catchment response. The results of these ongoing investigations will be reported in the future.

## APPENDIX A — OVERVIEW OF THE RAINFALL-RUNOFF MODEL

### *Mechanisms of storm runoff generation*

Hydrologic experience has revealed that runoff generation on catchments is a complex process. A number of mechanisms of storm runoff generation have been identified, chief among them are the infiltration excess and the saturation excess mechanisms (see Beven, 1986 for details).

The model described below incorporates both of these mechanisms and the effects of the spatial variability of soils, topography, and rainfall. The model predicts runoff generation rates and volumes only. The routing of the generated runoff over hillslopes and along stream channels is not included in the model. Neither are the effects of overland flow run-on due to the accumulated water up-gradient running on to neighboring areas and contributing to the infiltration and/or accumulation there.

### *A1. Prediction of saturation excess runoff*

For saturation excess runoff generation, the model uses a modified and spatially distributed version of TOPMODEL described in the papers of Beven and Kirkby (1979). Only an outline of the model is described. For a more rigorous presentation of the theory of TOPMODEL, the reader is referred to Beven (1986) and to Sivapalan et al. (1987). The model assumes that at any point $i$ on a hillslope, the downslope saturated subsurface flow rate $q_i$ can be expressed as:

$$q_i = T_i \tan\beta \exp\left[-\frac{S_i}{m}\right] \tag{A1}$$

where $\tan\beta$ is the slope of the ground surface, $T_i$ is a local soil transmissivity, $S_i$ is the local initial soil moisture storage deficit, and $m$ is a parameter which is proportional to the rate of change of hydraulic conductivity with depth. Beven (1986) showed that eqn. (A1) can be related to a similar exponential decline in hydraulic conductivity with depth:

$$K_{sz} = K_{si} \exp\left[-fz\right] \tag{A2}$$

where $K_{si}$ is the hydraulic conductivity at the soil surface and $z$ is the depth below it. For soils for which (A2) holds, and conductivity at large depth is small, Beven (1986) showed that $T_i \approx K_{si}/f$, and $m = \Delta\theta/f$, where $\Delta\theta$ is the moisture content deficit below the saturation value.

The following development of the theory is a slight variation from that of Beven (1986). Under steady-state conditions due to an assumed spatially uniform recharge rate $r$ to the saturated zone:

$$q_i = ar \tag{A3}$$

where $a$ is the area drained per unit contour length at point $i$. Using (A1) and (A3):

$$S_i = -m \ln\left(\frac{ar}{T_i \tan\beta}\right) \tag{A4}$$

Integrating (A4) over the complete catchment area $A$ we obtain an expression for the mean areal storage deficit $\bar{S}$ as:

$$\bar{S} = \frac{1}{A}\int_A -m \ln\left(\frac{ar}{T_i \tan\beta}\right) \mathrm{d}A \tag{A5}$$

Eliminating $r$ from (A4) and (A5), we have:

$$S_i = \bar{S} + m\lambda - m \ln\left(\frac{a\,T_e}{T_i \tan\beta}\right] \tag{A6}$$

where:

$$\lambda = \frac{1}{A}\int_A \ln\left(\frac{a\,T_e}{T_i \tan\beta}\right) \mathrm{d}A \tag{A7}$$

and:

$$\ln T_e = \frac{1}{A}\int_A (\ln T_i)\,\mathrm{d}A \tag{A8}$$

Given a knowledge of $\bar{S}$ and the spatial pattern of values of the topography-soil index $\ln(aT_e/T_i \tan\beta)$, equation (A6) will enable the prediction of the pattern of local initial storage deficits $S_i$ for all points on the catchment. The initial contributing area, which is the area for which $S_i < 0$ at the beginning of the storm, can also be predicted from (A6). Any rain that falls on this area immediately becomes saturation excess runoff. After the beginning of the storm some of the rain that falls on those points for which $S_i > 0$ infiltrates, thereby filling a portion of the soil moisture storage deficit. (The infiltration rates and volumes at these points are modeled by the equations given in Appendix A2.) In this case, the rainfall that is in excess of the storage deficit will add to the saturation excess runoff and the contributing area expands out from its initial position.

From equation (A6) it is clear that all points having identical values of $\ln(aT_e/T_i \tan\beta)$ are hydrologically similar with respect to saturation excess runoff generation. High values of $a$ and low values of $T_i$ and $\tan\beta$ will increase the likelihood of surface saturation.

All simulations reported in this paper used the following values of the parameters: $m = 3.0\,\mathrm{cm}$, $\bar{S} = 1.53\,\mathrm{cm}$. These values are typical values of comparable catchments.

*A2. Prediction of infiltration excess runoff*

For the infiltration excess runoff we use a model of point rainfall infiltration based on the following Philip (1957) equation:

$$g^* = c\,K_s + \tfrac{1}{2} S\, t^{-1/2} \tag{A9}$$

where $g^*$ is the infiltration capacity, $S$ is the sorptivity, and $K_s$ is the near surface saturated hydraulic conductivity. The sorptivity $S$ is related to $K_s$ as follows:

$$S = S_r\, K_s^{1/2} \tag{A10}$$

It is assumed that $K_s$ varies randomly in space but that $S_r$ and $c$ are constants.

The time to ponding $t_p$ and the post-ponding infiltration rate $g$ due to a rainfall of constant (in time) intensity $p$ can be shown to be:

$$t_p = \frac{S_r^2}{4\,c\,p}\left[\frac{p^2}{(p - c\,K_s)^2} - 1\right], \qquad p > K_s \tag{A11}$$

and:

$$g = c\,K_s + \tfrac{1}{2} S_r\, K_s^{1/2}(t - t_c)^{-1/2}, \qquad t > t_p \tag{A12}$$

respectively, where:

$$t_c = \frac{S_r^2\, K_s}{4\,p(p - c\,K_s)} \tag{A13}$$

The cumulative volumes of infiltration and runoff by infiltration excess at time $t$ after the beginning of the storm are given by:

$$\begin{aligned} G(t) &= pt, \qquad t < t_p \\ &= c\,K_s(t - t_c) + S_r K_s^{1/2}(t - t_c)^{1/2}, \qquad t > t_p \end{aligned} \tag{A14}$$

and:

$$Q(t) = pt - G(t) \tag{A15}$$

respectively. In this paper $S_r$ and $c$ are assigned values of $1.9837\,\mathrm{cm}^{1/2}$ and 0.667, respectively.

## APPENDIX B SPATIAL VARIABILITY OF $p$ AND $K_s$

Both the rainfall intensity $p$ and the soil hydraulic conductivity $K_s$ are assumed to vary randomly across the catchment. $K_s$ is assumed to follow a lognormal distribution with mean $\overline{K_s}$ and coefficient of variation $C_{vK}$. $K_s$ is also correlated in space and the correlation structure of $\ln K_s$ is assumed to be isotropic and to be given by a Bessel type correlation function.

The rainfall intensity $p$ is assumed to follow a two-parameter gamma distribution with mean $\bar{p}$ and coefficient of variation $C_{vp}$. The spatial correlation structure of the rainfall field is also assumed to be isotropic and to be given by (Sivapalan and Wood, 1987):

$$\rho_p(r) = a_1 \exp(-b_1^2 r^2) + a_2 \exp(-b_2^2 r^2) \tag{B1}$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are constants with $a_1 + a_2 = 1$.

The experiments reported in this paper all used the following parameter values for soils and rainfall: $\overline{K_s} = 0.008325\,\mathrm{cm\,min^{-1}}$, $C_{vK} = 1.0$; $\bar{p} = 0.008325\,\mathrm{cm\,min^{-1}}$, $C_{vp} = 1.6$, $a_1 = 0.7$, $a_2 = 0.3$, $b_1/b_2 = 3.0$.

## APPENDIX C SPATIAL VARIABILITY OF INFILTRATION, ANALYTICAL RESULTS

Analytical expressions were derived for the statistics of storm response due to spatially variable soils and rainfall. The derivations utilize the ponding time formula, eqn. (A11), and the point equations for infiltration and runoff, eqns. (A12)–(A15). Using eqn. (A11) an analytical expression was derived for $\alpha(t|p)$, the proportion of the catchment area that is saturated at time $t$ due to spatially variable soils and uniform rainfall of intensity $p$, as:

$$\alpha(t|p) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}\left\{\frac{\ln h_1(t) - \overline{K_s}}{\sqrt{2}\, C_{vK}\overline{K_s}}\right\} \tag{C1}$$

where $h_1(t)$ is the inverse of the ponding time formula (A11).

Once $\alpha(t|p)$ is known, the proportion $\alpha(t)$ of the catchment area that is saturated when both rainfall and soil are spatially variable is calculated. Noting that $\alpha(t|p)$ is a conditional value, dependent on $p$, $\alpha(t)$ can be expressed as:

$$\alpha(t) = \int_0^\infty \alpha(t|p) f_P(p)\mathrm{d}p \tag{C2}$$

The cumulative runoff volume is:

$$F_q = F_p - F_g \tag{C3}$$

Here $F_p$, the cumulative rainfall volume is just $F_p = \bar{p}\,t$, where $\bar{p}$ is mean areal rainfall rate and $t$ is storm duration. $F_g$ is the cumulative infiltration volume and its expected value is given by:

$$\begin{aligned} F_g(t) = {} & (1-\alpha)\bar{p}^{(1-\alpha)}t + c\,\alpha\overline{K_s}^{\alpha}\left[t - \frac{S_r^2\overline{K_s}^{\alpha}}{4\,\bar{p}^{\alpha}(\bar{p}^{\alpha} - c\,\overline{K_s}^{\alpha})}\right] \\ & + S_r\,\alpha\overline{K_s}^{1/2\alpha}\left[t - \frac{S_r^2\overline{K_s}^{\alpha}}{4\,\bar{p}^{\alpha}(\bar{p}^{\alpha} - c\,\overline{K_s}^{\alpha})}\right]^{1/2} \end{aligned} \tag{C4}$$

where:

$$\overline{p^{l(1-\alpha)}} = \int_0^\infty p^l f_P(p|t_p > t)\mathrm{d}p$$

$$\overline{p^{m\alpha}} = \int_0^\infty p^m f_P(p|t_p > t)\mathrm{d}p \tag{C5}$$

and:

$$\overline{K_s^{n\alpha}} = \int_0^\infty K_s^n f_K(K_s|t_p > t)\mathrm{d}K_s$$

The mean runoff production rate for a storm event can be expressed as:

$$m_q(t) = \bar{p} - m_g(t) \tag{C6}$$

where $m_g(t)$ is the mean infiltration rate. This was derived as:

$$m_g(t) = (1-\alpha)\bar{p}^{(1-\alpha)} + c\,\alpha\,\overline{K_s^\alpha} + \frac{S_r\,\alpha\,\overline{K_s^{1/2\alpha}}}{2\left[t - \dfrac{S_r^2\overline{K_s^\alpha}}{4\bar{p}^\alpha(\bar{p}^\alpha - c\,\overline{K_s^\alpha})}\right]^{1/2}} \tag{C7}$$

ACKNOWLEDGEMENTS

This work was supported in part by NASA grant number NAG 5-491 and USGS grant number 14-08-0001-G1138. This support is gratefully acknowledged.

REFERENCES

Amorocho, J., 1961. Discussion on "Predicting storm runoff on small experimental watershed", by N.E. Minshall. J. Hydraul. Div., Am. Soc. Civ. Eng., 87 (HY2): 185–191.

Band, L. and Wood, E.F., 1986. Computer graphics for distributed hydrologic modeling. Paper No. H22B-03. Am. Geophys. Union, Spring Meet., Baltimore, Md.

Bear, J., 1972. Dynamics of Fluids in Porous Media. Elsevier, Amsterdam, 764 pp.

Beven, K.J., 1986. Runoff production and flood frequency in catchments of order $n$: an alternative approach. In: V.K. Gupta, I. Rodriguez-Iturbe and E.F. Wood (Edtiors), Scale Problems in Hydrology. Reidel, Dordrecht, pp. 107–132.

Beven, K.J. and Kirkby, M.J., 1979. A physically based variable contributing area model of basin hydrology. Hydrol. Sci. Bull., 24 (1): 43–69.

Cushman, J.H., 1984. Fourier representation of multiphase averaging theory. Adv. Water Res., 7: 126–131.

Dagan, G., 1986. Statistical theory of groundwater flow and transport: Pore to laboratory, laboratory to formation, and formation to regional scale. Water Resour. Res., (9): 120s–134s.

Dooge, J.C.I., 1982. Parameterization of hydrologic processes. In: P.S. Eagleson (Editor). Proceedings of the Greenbelt Study Conference. Cambridge University Press, New York, N.Y., pp. 243–288.

Gupta, V.K., Waymire, E. and Rodriguez-Iturbe, I., 1986. On scales, gravity, and network structure in basin runoff. In: V.K. Gupta, I. Rodriguez-Iturbe and E.F. Wood (Editors), Scale Problems in Hydrology. Reidel, Dordrecht, pp. 159–184.

Hazzanizadeh, M. and Gray, W.G., 1979. General averaging equations for multi-phase systems: 1. Averaging procedure. Adv. Water Res., 2: 131–144.

Lumley, J.L. and Panofsky, H.A., 1964. The structure of Atmospheric Turbulence. Wiley, New York, N.Y., 240 pp.

Minshall, N.E., 1960. Predicting storm runoff on small experimental watersheds. J. Hydraul. Div., Am. Soc. Civ. Eng., 86(HY8): 17–38.

Philip, J.R., 1957. The theory of infiltration, 1–7. Soil Sci., Vols. 83, 84 and 85.

Shapiro, A.M., 1981. Fractured porous media: Equation development and parameter identification. Ph.D. Diss. Dep. Civ. Eng., Princeton University, Princeton, N.J.

Sivapalan, M., 1986. Scale problems in rainfall, infiltration, and runoff production. Ph.D. Diss. Dep. Civ. Eng., Princeton University, Princeton, N.J.

Sivapalan, M. and Wood, E.F., 1987. A multi-dimensional model of nonstationary space–time rainfall at the catchment scale. Water Resour. Res., 23(7): 1289–1299.

Sivapalan, M., Beven, K.J. and Wood, E.F., 1987. On hydrological similarity: 2. A scaled model of storm runoff production. Water Resour. Res., 23(12): 2266–2278.